Strain-based multiaxial fatigue damage modelling

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ABSTRACT

A new multiaxial fatigue damage model named characteristic plane approach is proposed in this paper, in which the strain components are used to correlate with the fatigue damage. The characteristic plane is defined as a material plane on which the complex three-dimensional (3D) fatigue problem can be approximated using the plane strain components. Compared with most available critical plane-based models for multiaxial fatigue problem, the physical basis of the characteristic plane does not rely on the observations of the fatigue crack in the proposed model. The cracking information is not required for multiaxial fatigue analysis, and the proposed model can automatically adapt for different failure modes, such as shear or tensile-dominated failure. Mean stress effect is also included in the proposed model by a correction factor. The life predictions of the proposed fatigue damage model under constant amplitude loading are compared with a wide range of metal fatigue results in the literature.

Keywords  characteristic plane; critical plane; life prediction; multiaxial fatigue.

NOMENCLATURE

\( A, B = \) material parameters in the energy fatigue criteria
\( E = \) Young’s modulus
\( F_{NP} = \) non-proportionality factor
\( k', \beta = \) material parameters in the strain fatigue criteria
\( N_f = \) number of cycles to failure
\( n' = \) cyclic strength exponent
\( s = \) ratio of shear fatigue limit and tensile fatigue limit
\( \varepsilon_{a,c}, \gamma_{a,c} = \) normal shear strain amplitude, shear strain amplitude acting on the characteristic plane, respectively
\( \sigma_{a,c}, \tau_{a,c} = \) normal stress amplitude, shear stress amplitude acting on the characteristic plane, respectively
\( \varepsilon^H_{a,c}, \sigma^H_{a,c} = \) hydrostatic strain amplitude and hydrostatic stress amplitude, respectively
\( \varepsilon_{−1}, \tau_{−1} = \) uniaxial and torsional fatigue strain limits, respectively
\( \sigma_{−1}, \tau_{−1} = \) uniaxial and torsional fatigue stress limits, respectively
\( \varepsilon_{N_f}, \gamma_{N_f} = \) fatigue strengths at finite life \( N_f \) for uniaxial and torsional loadings, respectively
\( \varepsilon_{a,\text{NP}} = \) equivalent strain quantity considering the non-proportional correction
\( \varepsilon_{ae}, \varepsilon_{ap} = \) elastic and plastic strain amplitude, respectively

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\[ \rho = \text{material constant} \]
\[ \sigma_y = \text{yield strength} \]
\[ \sigma_{m,c} = \text{mean stress on the characteristic plane} \]
\[ \alpha = \text{angle between the characteristic plane and the maximum normal strain amplitude plane} \]

**INTRODUCTION**

Many critical mechanical components experience multiaxial cyclic loading during their service life. In recent decades, numerous attempts to develop multiaxial fatigue damage modelling have been reported. Several reviews and comparisons of existing multiaxial fatigue models can be found elsewhere.\(^1\)–\(^6\) However, the effectiveness of individual methods varies with materials, fracture mechanism and loading conditions.\(^7\) No single theory has been applied to a wide variety of materials and loading conditions.\(^8\) To the authors’ knowledge, no existing multiaxial fatigue damage model is universally accepted. This paper focuses on the strain-based models and thus only those models are briefly described below.

Early investigations used the equivalent strain approach\(^9\) and plastic work or plastic strain energy approach.\(^10\),\(^11\) In recent years, fatigue models based on the critical plane approach for multiaxial fatigue evaluation have been gaining popularity due to their success in accurately predicting lives.\(^3\),\(^12\) A number of the critical plane approaches that are based on strain or energy are for the shear failure mode.\(^13\)–\(^20\) Some other models are based on tensile failure mode.\(^8\),\(^21\)–\(^23\) It has been found that the methods based on one failure mode perform poorly for the fatigue modelling of the other failure mode.\(^8\),\(^20\),\(^23\) Bannantine and Socie\(^24\) suggest using two different models for different failure modes and choosing the better prediction as the final result. Similar methodologies are used by other researchers.\(^25\) Park and Nelson\(^26\) reviewed the two-model methodology suggested by Socie\(^8\) and stated that the failure modes depend on the materials. It appears that the failure mode depends not only on the material properties but also on the stress state.\(^8\),\(^27\) Lee et al.\(^27\) found that STS304 shows different failure modes in low- and high-cycle regime. This type of observation makes the two-model approach somewhat difficult to apply because no information is available for choosing the model before the failure modes are observed. Sometimes, the crack information is not available as it may not be possible due to the nature of the experimental design.\(^7\),\(^28\)

The critical plane approach was originally proposed based on the observations that the fatigue crack nucleation occurs at the persistent slip bands, formed in some grains (crystals) of the materials. The planes are named critical plane and the stress (or strain) components on it are used for fatigue analysis.\(^29\) That is the physical basis of the critical plane approach. This assumption or basis makes it difficult to apply the model to materials with microstructures different with normally used metals. Also, this assumption usually requires cracking analysis to distinguish the failure modes before the appropriate critical plane-based model can be applied.\(^17\) If both failure modes occur and neither of them dominates in the experiments, the decision of choosing the appropriate model is hard to make.

In this paper, a new fatigue damage model named characteristic approach is proposed. The characteristic plane approach is similar to the critical plane in the calculation procedure. A plane is first determined and the strain components on the plane are combined together and used for fatigue life prediction. Unlike most of existing critical plane-based models, the characteristic plane in the proposed model is not based on the physical observations of the crack but arises from the idea of dimension reduction. It assumes that the complex multiaxial fatigue problem can be approximated by using the strain components on a certain plane (named characteristic plane in this paper). Then the objectives are to find the plane and the formula of combinations of the strain components on that plane. Through this type of definition of the characteristic plane, failure mode analysis is not required and the proposed model can automatically adapt for different materials. Also, this definition gives the proposed model the potential to apply to materials with non-metal microstructures. The correction factors for the extra out-of phase plastic hardening and mean stress are also introduced to the proposed model. A wide range of experimental observations for metals available in the literature is used to validate the proposed model. Very good correlations are found between predicted and experimental fatigue lives under proportional and non-proportional loading for both low-cycle regime and high-cycle regime.

**PROPOSED MULTIAXIAL FATIGUE MODEL**

**Characteristic plane definition**

Multiaxial fatigue problem is complex because it usually involves three-dimensional (3D) stress (strain) histories. To directly analyse that the multiaxial fatigue problem is either cumbersome or unpractical, as it may require too much computational and experimental effort. Moreover, when the applied multiaxial loadings are
non-proportional, the determination of the stress (or strain) amplitudes becomes difficult. In this paper, we are trying to reduce the dimension of the problem and approximate the complex 3D fatigue problem by using the strain components on a certain plane, which reduce the dimension of the problem and simplified the calculation. This plane is named characteristic plane in the proposed model.

In the proposed model, we make an assumption that there exists a characteristic plane on which the strain components or their combinations can be used to approximate the complex multiaxial fatigue problem. Following this assumption, the two objectives of this paper become clear. One objective is to find the characteristic plane for different materials. The other is to find the formula for the combination of those strain components acting on the plane. The procedures are discussed in next following sections.

Before we proceed to the detailed derivation of the proposed method, a short discussion is given here to distinguish the difference between the characteristic plane-based model and the critical plane-based model. These two methods differed in three main aspects as described below.

1. Their physical bases are different. As described in the previous section, the critical plane approach originates from the observations of the fatigue crack, which is usually either the maximum normal stress (strain) plane (Mode I) or the maximum shear stress (strain) plane (Mode II or III). The characteristic plane approach originates from the dimension reduction idea, in which the main objective is to reduce the complexity of the multiaxial fatigue problem. The resulting characteristic plane is only a material plane on which the fatigue damage is evaluated. It may or may not have a direct relation with the fatigue crack orientation observed in the experiments. The physical difference makes the characteristic plane approach not require failure modes analysis before application to multiaxial fatigue damage calculation, which is usually required by the critical plane-based models. Also, the characteristic plane approach has the potential to apply to the non-metals, in which the non-crystal like microstructure violates the physical basis of the critical plane approach.

2. The identification procedures of the characteristic plane and the critical plane are different. Once the material failure mode is observed, the identification of the critical plane is straightforward. It only relies on stress (or strain) analysis. For different materials with the same failure mode, the critical plane orientation is fixed. It is either the maximum normal stress (or strain) plane or the maximum shear stress (or strain) plane. The characteristic plane in the proposed model is determined through minimizing the contributions of the hydrostatic strain amplitude to zero (as shown in General Case for Different Failure Modes). It explicitly relies both on the material properties and on the strain analysis. For different materials with the same failure mode, the characteristic plane could be different because it depends on both the uniaxial and the pure torsional $e-N$ curves. From this point of view, the determination of the critical plane is semi-analytical because it requires the analyst to determine the failure modes from experimental data or assume from experience. The characteristic plane determination is fully analytical because it only requires the quantitative data from uniaxial and torsional experiments.

3. The results and robustness of the characteristic plane approach and the critical plane approach are different. The result of the critical plane is a discrete function, which is either maximum normal strain plane or maximum shear strain plane. The result of the characteristic plane is a quantitative and continuous function. In the multiaxial fatigue experiments, usually both Mode I and Mode II cracks exist. For example, under pure shear tests, the crack usually occurs along the maximum shear strain plane and then propagates along the maximum principle stress plane. In that case, only visual or empirical observation is not good enough to decide which model to use. Also, if you make a decision based on a certain parameter exceeding a threshold value (e.g., Life of Mode II crack exceeds 70% of the total life), there is still a problem because you create a discontinuity subjectively. The material with 69% uses the critical plane of maximum normal strain amplitude and the material with 71% uses the critical plane, which is 45 degree off the maximum normal strain amplitude plane. Therefore, a quantitative and continuously varying model is more desirable. For the material changing failure modes with respect to loadings and environmental conditions or the material without failure mode information, it is risky to apply either of those models because their error is unpredictable. From this point of view, the proposed model is more robust because it can automatically adapt to those conditions.

**Fatigue damage parameter**

In this section, a new damage parameter defined on the characteristic plane is proposed. Consider the fully reversed uniaxial-torsional fatigue problem (with no mean stress). The strain tensors under plane stress condition are given in Eq. (1):

\[
\begin{bmatrix}
\varepsilon_a & \gamma_a/2 & 0 \\
\gamma_a/2 & -\nu_{eff}\varepsilon_a & 0 \\
0 & 0 & -\nu_{eff}\varepsilon_a
\end{bmatrix},
\]

where $\varepsilon_a$ and $\gamma_a$ are the normal and shear strain amplitude (half of the strain range), respectively. $\nu_{eff}$ is the effective...
Poisson’s ratio which is given by

\[ \varepsilon_{\text{eff}} = \frac{\varepsilon_{ae} + H_{ap}}{\varepsilon_{ae}}, \]  

where \( \varepsilon_c \) is the elastic Poisson’s ratio. If no experimental value is available, a value of 0.3 can be used instead. \( \varepsilon_p \) is the plastic Poisson’s ratio and takes the value of 0.5. \( \varepsilon_{ae} \) and \( \varepsilon_{ap} \) are the elastic and plastic strain amplitude, respectively. They can be calculated from a cyclic stress–strain relationship, such as the Ramberg–Osgood equation:

\[ \varepsilon_a = \varepsilon_{ae} + \varepsilon_{ap} = \frac{\sigma_a}{E} + \left( \frac{\sigma_a}{K^*} \right)^{\frac{1}{n}} \]

where \( E \) is the Young’s modulus. \( K^* \) and \( n^* \) are the cyclic strength coefficient and the cyclic strength exponent, respectively.

To simplify the discussion, we first investigate the fatigue failure criteria at the fatigue limit stage. As stated in the previous section, one objective in the proposed method is to find an appropriate formula to combine the strain components for fatigue damage evaluation. It is obvious that both shear stress (or strain) and normal stress (or strain) contribute to the final failure of mechanical components under multiaxial fatigue loading. However, several researchers have also noticed the importance of hydrostatic stress and included its effect in their models. It was also shown that the contribution of hydrostatic stress is different for different models and seems to vary with materials. The proposed model includes damage contribution from three sources—the normal strain \( (\varepsilon_{a,c}) \), shear strain acting \( (\gamma_{a,c}) \) on the characteristic plane and hydrostatic strain amplitude \( (\varepsilon_{aH}) \). It assumes that the material fails when the summation of the normalized energies due to the three strain components reaches a critical value. A mathematical expression is formulated as

\[ \frac{\varepsilon_{a,c} \sigma_{a,c}}{\varepsilon_{-1} \sigma_{-1}} + \frac{\gamma_{a,c} \tau_{a,c}}{\gamma_{-1} \tau_{-1}} + A \frac{\varepsilon_{aH}^2 \sigma_{aH}}{\varepsilon_{-1} \sigma_{-1}} = B, \]

where \( \sigma_{a,c} \), \( \tau_{a,c} \), and \( \sigma_{aH} \) are the normal stress amplitude, shear stress amplitude acting on the characteristic plane and hydrostatic stress amplitude, respectively. \( \varepsilon_{-1} \) and \( \gamma_{-1} \) are uniaxial and torsional fatigue strain limits, respectively. \( \sigma_{-1} \) and \( \tau_{-1} \) are uniaxial and torsional fatigue stress limits, respectively. \( A \) and \( B \) are material parameters which are determined from uniaxial and torsional fatigue tests.

The physical meaning of Eq. (4) is that the final damage is the summation of the damage caused by different energy components. Parameter \( A \) is a material parameter and considers the different contributions of the hydrostatic strain amplitude corresponding to different materials. Under the fatigue limit state, the material is usually elastic. In that case, Eq. (4) can be simplified only using strain terms.

\[ \sqrt{\left( \frac{\varepsilon_{a,c}}{\varepsilon_{-1}} \right)^2 + \left( \frac{\gamma_{a,c}}{\gamma_{-1}} \right)^2 + k \left( \frac{\varepsilon_{aH}}{\varepsilon_{-1}} \right)^2} = \beta, \]

where \( k \) and \( \beta \) are material parameters which can be determined by uniaxial and torsional fatigue tests. The strain-based version is easy to calculate compared with the model directly using energy terms. Also, the fatigue properties expressed in \( e-N \) curves are in common use already. This makes it easy to implement the proposed method to practical applications.

Case 1: Tensile failure mode

Because the orientation of the characteristic plane has not been determined yet, suppose that for one type of material the characteristic plane coincides with the maximum normal strain amplitude plane. In this case, the characteristic plane is similar with the critical plane definition of tensile failure mode.

For a fully reversed uniaxial fatigue experiment \( (\varepsilon_a = \varepsilon_{-1}, \gamma_a = 0) \), the strain components on the characteristic plane are

\[ \begin{align*}
\varepsilon_{a,c} &= \varepsilon_{-1} \\
\gamma_{a,c} &= 0 \\
\varepsilon_{aH} &= \varepsilon_{-1}(1 - 2\varepsilon_{eff})/3.
\end{align*} \]

For a fully reversed pure torsional fatigue experiment \( (\varepsilon_a = 0, \gamma_a = \gamma_{-1}) \), the strain components on the characteristic plane are

\[ \begin{align*}
\varepsilon_{a,c} &= \gamma_{-1}/2 \\
\gamma_{a,c} &= 0 \\
\varepsilon_{aH} &= 0.
\end{align*} \]

Substitute Eqs (6) and (7) by Eq. (5) and solve for material parameters to obtain

\[ \begin{align*}
\beta &= \frac{1}{2} \frac{\gamma_{-1}}{\varepsilon_{-1}} \\
k &= \left[ \frac{1}{4} \left( \frac{\gamma_{-1}}{\varepsilon_{-1}} \right)^2 - 1 \right] \frac{9}{(1 - 2\varepsilon_{eff})^2}. \end{align*} \]

Notice that the physical meaning of \( k \) is the contribution of damage caused by the hydrostatic strain amplitude. It should be non-negative. So \( \gamma_{-1}/\varepsilon_{-1} \) should not be less than two. Recall the assumption made before this calculation. It is only possible for a material \( (\gamma_{-1}/\varepsilon_{-1} \geq 2) \) that the characteristic plane coincides with the maximum normal strain plane using the present damage parameter [Eq. (5)]. It is also interesting to note that \( k \) equals zero.
when \( \gamma_{-1}/\varepsilon_{-1} = 2 \), which means [from Eq. (5)] that the hydrostatic strain amplitude has no contribution to the fatigue damage for this material according to the present definition of the damage parameter (Eq. (5)).

**Case 2: Shear failure mode**

Now suppose that for one type of material, the characteristic plane is 45 degrees off the maximum normal strain amplitude plane, which is the maximum shear strain amplitude plane for uniaxial and torsional loading. In this case, the characteristic plane is similar to the critical plane of shear failure mode. Following the steps described above, the material parameters \( k \) and \( \beta \) are once again calculated.

For a fully reversed uniaxial fatigue experiment (\( \varepsilon_a = \varepsilon_{-1}, \gamma_a = 0 \)), the strain components on the characteristic plane are

\[
\begin{align*}
\varepsilon_{a,c} &= \varepsilon_{-1}(1 + \upsilon_{\text{eff}})/2, \\
\gamma_{a,c} &= \varepsilon_{-1}(1 + \upsilon_{\text{eff}})/2, \\
\varepsilon_{a}^{H} &= \varepsilon_{-1}(1 - 2\upsilon_{\text{eff}})/3.
\end{align*}
\]

(9)

For a fully reversed pure torsional fatigue experiment (\( \varepsilon_a = 0, \gamma_a = \gamma_{-1} \)), the strain components on the characteristic plane are

\[
\begin{align*}
\varepsilon_{a,c} &= 0, \\
\gamma_{a,c} &= \gamma_{-1}, \\
\varepsilon_{a}^{H} &= 0.
\end{align*}
\]

(10)

Substituting Eqs (9) and (10) in Eq. (5) and solving for the material parameters:

\[
\begin{align*}
\beta &= 1, \\
 k &= \left[ 1 - \frac{1}{4}(1 - \upsilon_{\text{eff}}) - (1 + \upsilon_{\text{eff}})^2 \left( \frac{\gamma_{-1}}{\varepsilon_{-1}} \right)^2 \right] \left( 1 - 2\upsilon_{\text{eff}} \right)^2.
\end{align*}
\]

(11)

From Eq. (11), \( \gamma_{-1}/\varepsilon_{-1} \) should not be less than \( 2\sqrt{\frac{1+\upsilon_{\text{eff}}}{1-\upsilon_{\text{eff}}}} \). The mechanical component is usually kept in elastic condition near the fatigue limit regime, thus \( \upsilon_{\text{eff}} \) can be approximated using 0.3. Under this assumption, \( \gamma_{-1}/\varepsilon_{-1} \) should not be less than 1.39. Thus, it is only possible for a certain type of material \( \{ \gamma_{-1}/\varepsilon_{-1} \geq 2\sqrt{\frac{1+\upsilon_{\text{eff}}}{1-\upsilon_{\text{eff}}}} \} \) that the characteristic plane could be the maximum shear strain amplitude plane using the present damage parameter [Eq. (5)]. Similar to the first case, \( k \) equals zero when \( \gamma_{-1}/\varepsilon_{-1} = 2\sqrt{\frac{1+\upsilon_{\text{eff}}}{1-\upsilon_{\text{eff}}}} \), which means [from Eq. (5)] that the hydrostatic strain amplitude \( \varepsilon_{a}^{H} \) has no contribution to the fatigue damage for this material according to the present definition of the damage parameter.

Several conclusions can be drawn based on the derivations of the characteristic plane orientations for the two cases above. The contribution of the hydrostatic strain amplitude is different for different materials if the characteristic plane is fixed for all the materials. There are two special types of material \( \{ \gamma_{-1}/\varepsilon_{-1} = 2 \} \) and \( \{ \gamma_{-1}/\varepsilon_{-1} = 2\sqrt{\frac{1+\upsilon_{\text{eff}}}{1-\upsilon_{\text{eff}}}} \} \) for which the contribution of the hydrostatic strain amplitude is zero if the characteristic plane is defined as shown in Cases 1 and 2. It is also noticed that, if the characteristic plane is fixed, the applicability of individual model in Cases 1 and 2 is limited.

**General case for different failure modes**

Instead of fixing the characteristic plane, the current model searches for the characteristic plane orientations on which the contribution of the hydrostatic strain amplitude is minimized to zero. This is the general approach that can be applied to all materials.

For an arbitrary material, let the angle between the characteristic plane and the maximum normal strain amplitude plane be \( \alpha \). Because the contribution of the hydrostatic strain amplitude is zero, Eq. (5) is rewritten as

\[
\sqrt{\left( \frac{\varepsilon_{a,c}}{\varepsilon_{-1}} \right)^2} + \left( \frac{\gamma_{a,c}}{\gamma_{-1}} \right)^2 = \beta.
\]

(12)

The objective is to find \( \alpha \) and \( \beta \) for an arbitrary material, following the steps described for the first two cases.

For a fully reversed uniaxial fatigue experiment (\( \varepsilon_a = \varepsilon_{-1}, \gamma_a = 0 \)), the strain components on the characteristic plane are given as

\[
\begin{align*}
\varepsilon_{a,c} &= \frac{(1 - \upsilon_{\text{eff}})}{2} \varepsilon_{-1} + \frac{(1 + \upsilon_{\text{eff}})}{2} \varepsilon_{-1} \cos(2\alpha), \\
\gamma_{a,c} &= (1 + \upsilon_{\text{eff}}) \varepsilon_{-1} \sin(2\alpha).
\end{align*}
\]

(13)

For a fully reversed pure torsional fatigue experiment (\( \varepsilon_a = 0, \gamma_a = \gamma_{-1} \)), the strain components on the characteristic plane are given as

\[
\begin{align*}
\varepsilon_{a,c} &= \frac{\gamma_{-1}}{2} \cos(2\alpha), \\
\gamma_{a,c} &= \gamma_{-1} \sin(2\alpha).
\end{align*}
\]

(14)

Substituting Eqs (13) and (14) in Eq. (12) and solving for \( \alpha \) and \( \beta \),
where \( s = \frac{s_1}{s_2} \) is a material constant. Here \( \alpha \) takes values from 0 to \( \frac{\pi}{4} \). As shown in Eq. (15), both \( \alpha \) and \( \beta \) are the functions of the material property \( s \) and the effective Poisson’s ratio \( v_{\text{eff}} \). It is found that \( \alpha \) increases as \( s \) decreases. \( \alpha \) equals 0 when \( s \) equals 2, and \( \alpha \) equals \( \frac{\pi}{4} \) when \( s \) equals \( 2 \sqrt{\frac{1 + v_{\text{eff}}^2}{1 - v_{\text{eff}}^2}} \). \( \beta \) does not change monotonically with respect to \( s \), but all the \( \beta \) values are close to 1 for materials with \( 2 \sqrt{\frac{1 + v_{\text{eff}}^2}{1 - v_{\text{eff}}^2}} < s < 2 \). The variations of \( \alpha \) and \( \beta \) with respect to \( s \) are plotted in Fig. 1 with two different Poisson’s ratios. One is for pure elastic condition and takes the value of 0.3. The other is for pure plastic condition and takes the value of 0.5. Note that the physical meaning of \( \alpha \) is an indication of material failure mode. When \( \alpha \) is close to 0 degree, it indicates the tensile failure mode. When \( \alpha \) is close to 45 degree, it indicates the shear failure mode. As shown in Fig. 1, the value of \( \alpha \) also depends on the effective Poisson’s ratio. It appears that, for the material with constant value of \( s \), shear failure mode is likely to occur under low-cycle regime (larger plastic deformation).

From Eq. (15), \( \alpha \) has no real solution for \( s > 2 \). This indicates that for those materials, such as 5% chrome work roll steel,\(^{23}\) the contribution of hydrostatic strain amplitude cannot be minimized to 0 and must be considered during the fatigue damage evaluation. We use the results in Case 1 for the materials of \( s > 2 \). The summary of material parameters for all types of materials are listed in Table 1.

**Fatigue life model**

After developing the fatigue limit criterion as above, the methodology for fatigue life prediction is relatively easy. Notice that the fatigue limit often refers to the fatigue strength at very high cycle (usually \( 10^6 \) to \( 10^7 \) cycles). For finite fatigue life prediction, the damage parameter should be correlated with the life (number of loading cycles). Equation (5) can be rewritten as

\[
\frac{1}{\beta} \sqrt{(\varepsilon_a c)^2 + \frac{(\varepsilon_{c-1})^2(\gamma_a c)^2}{\gamma_{c-1}} + k(\varepsilon_H^c)^2} = \varepsilon_{c-1}.
\]

The left-hand side of Eq. (16) can be treated as the equivalent strain amplitude and can be used to correlate with the fatigue life using the uniaxial \( e-N \) curve. Thus the fatigue life model is expressed as

\[
\frac{1}{\beta} \sqrt{(\varepsilon_a c)^2 + \frac{(\varepsilon_{N_c})^2(\gamma_a c)^2}{\gamma_{N_c}} + k(\varepsilon_H^c)^2} = \varepsilon_{N_c} = f(N_c).
\]

![Fig. 1 Variations of \( \alpha \) and \( \beta \) at different Poisson’s ratios.](image)

**Table 1** Material parameters for fatigue damage evaluation

<table>
<thead>
<tr>
<th>Material parameters</th>
<th>( s = \frac{s_1}{s_2} \leq 2 )</th>
<th>( s = \frac{s_1}{s_2} &gt; 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos(2\alpha) )</td>
<td>( \frac{-B + \sqrt{B^2 - 4AC}}{2A} )</td>
<td>( \frac{-B + \sqrt{B^2 - 4AC}}{2A} )</td>
</tr>
<tr>
<td>( A )</td>
<td>( 1 + v_{\text{eff}}^2 + 1 - \frac{4(1 + v_{\text{eff}}^2)}{s^2} )</td>
<td>( 1 + v_{\text{eff}}^2 + 1 - \frac{4(1 + v_{\text{eff}}^2)}{s^2} )</td>
</tr>
<tr>
<td>( B )</td>
<td>( 2(1 - v_{\text{eff}}^2) )</td>
<td>( 2(1 - v_{\text{eff}}^2) )</td>
</tr>
<tr>
<td>( C )</td>
<td>( 1 - v_{\text{eff}}^2 + \frac{4(1 + v_{\text{eff}}^2)}{s^2} )</td>
<td>( 1 - v_{\text{eff}}^2 + \frac{4(1 + v_{\text{eff}}^2)}{s^2} )</td>
</tr>
<tr>
<td>( k )</td>
<td>( k = 0 )</td>
<td>( k = \frac{\frac{1}{4}s^2 - 1}{(1 - v_{\text{eff}}^2)} )</td>
</tr>
<tr>
<td>( \beta )</td>
<td>( \beta = \frac{1}{4}\cos^2(2\alpha)s^2 + \sin^2(2\alpha) )</td>
<td>( \beta = \frac{1}{4}s^2 + \sin^2(2\alpha) )</td>
</tr>
</tbody>
</table>
where $N_f$ is the number of cycles to failure and $f(N_f)$ is the uniaxial strain-life function obtained from experimental results. Notice here $\varepsilon_{-1}$ and $\gamma_{-1}$ in Eq. (16) are replaced by $\varepsilon_N$ and $\gamma_N$, respectively, which are fatigue strengths at finite life $N_f$ for uniaxial and torsional loadings. Equation (17) has no closed form solution. In practical calculation, a trial and error method can be used to find $N_f$, $\varepsilon_N$ and $\gamma_N$ take initial values as $\varepsilon_{-1}$ and $\gamma_{-1}$. It is found that usually a few iterations are enough to make $N_f$ converge. Equation (17) together with the parameters in Table 1 is used for fatigue life prediction in this paper. The quantity in Table 1 is redefined as $\varepsilon = \frac{\varepsilon_N}{\gamma_N}$.

**Out-of-phase hardening**

Under out-of-phase non-proportional loading, the principal stress and strain direction rotates during one cycle of loading. If plastic deformation occurs, it causes additional hardening of the material. Due to the additional hardening, the stress amplitude increases under the same applied strain amplitude for out-of-phase loading and thus reduces the fatigue life. A pure strain-based approach does not take into account the additional hardening because there are no stress terms. There are some methodologies to overcome this drawback. Socie included a stress term on the critical plane to consider the additional hardening caused by out-of-phase loading. The energy-based approach can consider this effect because the stress term is inherent in the energy expression. However, if the stress term is used, plasticity theory is required to predict more accurate elasto-plastic hysteresis loops under non-proportional loading. Although there are some available plasticity models for calculating the stress–strain relationship under non-proportional loading, these models usually require extensive numerical computational efforts and many material constants requiring several multi-axial experiments. For the engineering application, a simple correction factor was used to consider the additional hardening. The general form is given as

$$\varepsilon_{a,\text{NP}} = (1 + \rho F_{\text{NP}})\varepsilon_a,$$  \hspace{1cm} (18)

where $\varepsilon_{a,\text{NP}}$ is the equivalent strain quantity considering the non-proportional correction. $\varepsilon_a$ is the multiaxial fatigue strain parameter used by different models. $F_{\text{NP}}$ is the so-called non-proportionality factor and depends on different strain paths. $\rho$ is a material constant which indicates the material sensitivity to the out-of-phase loading. Different authors gave different definitions for $F_{\text{NP}}$ and $\rho$. Borodi and Strizhalo found that, with the same maximum strain amplitude, higher hardening occurs for the strain path which envelops a larger effective area. This phenomenon is also supported by numerous experimental data.

In this paper, a simple definition of $F_{\text{NP}}$ is suggested and the value of $\rho$ can be calibrated using one set of out-of-phase fatigue tests. First, the strain path is plotted in normalized coordinates, on which the $x$-axis represents the normal strain divided by maximum normal strain amplitude and the $y$-axis represents the shear strain divided by the maximum shear strain amplitude. The non-proportionality factor is defined as the envelope area divided by 4. Several strain paths used in this paper for model validations are plotted in Fig. 2.

In Fig. 2, the uniaxial, torsional and proportional loading paths have no envelope area, thus $F_{\text{NP}}$ equals to zero. For box path loading [Fig. 2 (ii)], $F_{\text{NP}}$ achieves the maximum value of 1. For other strain paths, $F_{\text{NP}}$ equals the shaded area divided by the box strain path area (dashed line). In this paper, only constant multiaxial loading is considered. For variable multiaxial fatigue loading, suitable modification may be needed. In the next section, it is shown that the current definition of $F_{\text{NP}}$ obtains very good predictions under constant non-proportional multiaxial loading for various metals.

One additional multiaxial data with a specific value of $F_{\text{NP}}$ is used to calibrate the material constant $\rho$ using Eq. (18). If the strain amplitude is low and the specimen is under elastic condition, additional hardening does not occur, and thus $\rho$ can take the value of zero. If no experimental data are available, a simple formula is suggested as Eq. (19), which is obtained from the experimental data collected in Section 3 (Table 2). The simplified function and the experimental value are plotted in Fig. 3.

$$\rho = \begin{cases} 0 & \varepsilon_a < \frac{\sigma_y}{E} \\ 0.55 + 0.45\cos(\pi(s - 1)) & \varepsilon_a \geq \frac{\sigma_y}{E} \end{cases}$$ \hspace{1cm} (19)

**Mean stress effect**

Practical mechanical components generally experience cyclic fatigue loading together with the mean stress. The mean stress could also be introduced by residual stress, environmental effects, etc. It is well known that the mean normal stress has an important effect on fatigue life. Normally, tensile mean stress reduces the fatigue life, while compressive mean stress increases the fatigue life.

There are many models for mean normal stress effect correction. Gerber, Goodman, Soderberg and Morrow proposed different correction factors. Kujawski and Ellyin proposed a unified approach to mean stress correction. For the multiaxial fatigue problem, mean normal stress is included in the model in different ways depending on different models. Fatemi and Socie considered the maximum normal stress acting on the critical
Fig. 2 Different strain paths used in this study.

plane. Papadopoulos\textsuperscript{4} considered the hydrostatic mean stress. Farahani\textsuperscript{20} used a correction factor based on the mean stress on the critical plane.

Based on the experimental data collected from the literature, the mean stress is introduced to the fatigue model in this paper by a correction factor similar to Soderberg\textsuperscript{43} as \(1 - \frac{\sigma_{m,c}}{\sigma_y}\). Thus, Eq. (5) is rewritten as

\[
(1 + \rho F_{NP}) \frac{1}{\beta} \sqrt{\left(\frac{\varepsilon_{a,c}}{\varepsilon_{a,c}}\right)^2 + \left(\frac{\varepsilon_{N}}{\varepsilon_{N}}\right)^2 + k(\varepsilon_{a,c})^2 = f(N)} ,
\]

where \(\sigma_{m,c}\) is the mean stress on the characteristic plane and \(\sigma_y\) is the yield strength of the material. Comparisons with the available experimental data in the next section show a good correlation using this correction factor. However, for an arbitrary material, the analyst could use different correction factors such as the one suggested by Kujawski and Ellyin\textsuperscript{45} and calibrate the factors using the experimental data.

In the proposed model [Eq. (20)], \(k\) and \(\beta\) are determined from experimental values of \(\varepsilon_{N}\) and \(\gamma_{N}\), which are obtained using uniaxial and torsional \(e-N\) curves. \(F_{NP}\) is the stain path parameter and not a material parameter, and thus does not require calibration using experimental data. \(\rho\) is the only fitting constant requiring calibration using one additional non-proportional loading experiment.

As a result of the above derivation, the methodology becomes very simple with the current model. For any arbitrary loading history, the maximum normal strain amplitude plane is identified. This is achieved by enumeration, by changing the angle by 1 degree increment. Then the angle \(\alpha\) and material parameters are determined for different materials according to Table 1. The characteristic plane is the plane, which has an angle \(\alpha\) with the maximum normal strain amplitude plane. Finally, the strain components on the characteristic plane are calculated and the fatigue damage is evaluated using Eq. (20). Note that the characteristic plane in the proposed model explicitly depends on both the strain state (maximum normal strain amplitude plane) and the material properties (angle \(\alpha\)).

**VALIDATION OF FATIGUE LIFE PREDICTION MODEL**

Twelve sets of fatigue experimental data are employed in this section, and are listed in Table 2. It is noted that the purpose of these comparisons is to validate the model’s generality to different materials and conditions. The collected data cover materials used in a lot of different industries, such as construction engineering, automotive engineering and aerospace engineering. They differ in several ways, such as failure mechanism (3 shear, 3 tensile, 2 mixed and 4 unknown) loading path and other unique characteristics (temperature, surface treatments, etc.). Table 2 includes a detailed description of the experimental data collected in this paper, such as material name, reference, multiaxial strain path, failure mode, \(s\) range, stain...
Table 2  Experimental data used for model validation

<table>
<thead>
<tr>
<th>Material</th>
<th>Refs</th>
<th>Strain amplitude range (%)</th>
<th>Failure mechanism</th>
<th>Other Information</th>
</tr>
</thead>
<tbody>
<tr>
<td>WASPALLOY 48, 49</td>
<td>Pro, sin90</td>
<td>1.3–1.7</td>
<td>Shear</td>
<td>Different surface treatments</td>
</tr>
<tr>
<td>AL-6061-T6</td>
<td>Pro</td>
<td>1.52–1.57</td>
<td>Mixed and shear dominated</td>
<td>Different temperatures</td>
</tr>
<tr>
<td>HASTELLOY at 23°C</td>
<td>Pro, sin90, cro, ball</td>
<td>2.3–2.7</td>
<td>Tensile</td>
<td>At the temperature of 760°C</td>
</tr>
<tr>
<td>TI-6Al-4V alloy</td>
<td>Pro, sin90, tr</td>
<td>2.2–2.5</td>
<td>Mixed and shear dominated</td>
<td>Mean stress included</td>
</tr>
<tr>
<td>Haynes 188</td>
<td>Pro, sin90, tr</td>
<td>2.2–2.5</td>
<td>Mixed and shear dominated</td>
<td>Mean stress included</td>
</tr>
</tbody>
</table>

As mentioned in ‘General Case for Different Failure Modes’, the proposed method shows a positive correlation between the ratio $s$ and material failure mechanism. Shear-dominated failure is likely to occur for a low value of $s$, and tensile-dominated failure is likely to occur for a high value of $s$. Experimental observations are also found to support this statement. Table 2 includes the failure patterns observed in the experiments and also the $s$ values defined in the proposed model. It shows a positive correlation between the failure mechanisms and the $s$ value. For shear-dominated failure (A533B pressure vessel steel, Waspaloy, Al-6061-T6), lower values of $s$ are observed and range from 1.52 to 1.7. For tensile-dominated failure (5% chrome work roll steel, 304 stainless steel, 45 steel), higher values of $s$ are observed and range from 2.3 to 3.1. For mixed failure (1Cr–18Ni–9Ti stainless steel, Hastelloy-X at different temperatures), moderate values of $s$ are observed and range from 1.4 to 2.7. This type of observations indicates the material failure patterns can be related or explained by the $s$ values as suggested by the proposed method.

The predicted fatigue lives and the experimental lives are plotted together in Fig. 4. In Fig. 4, the $x$-axis is the experimental life and the $y$-axis is the predicted life. Both lives are in log scale. The solid line indicates that the predicted results are identical with experimental results. The dashed lines are the life factor of 2. The different strain paths shown in the legend are also shown in Fig. 2.

As shown in the Fig. 4, the predicted results agree with the experimental results very well. Most of the points fall within the life factor of 2. The worst case of the proposed model's life prediction is for 5% chrome work roll steel. As mentioned by Kim et al., there is more scatter in life data than usually observed in the laboratory for ductile metals. This is believed to be an inherent characteristic of materials whose life is controlled by defects. Despite the larger scatter, the proposed model predicts the trend very well.
CONCLUSIONS

A new multiaxial fatigue life prediction model is proposed in this paper for constant amplitude, in-phase and out-of-phase loading conditions. Twelve sets of experimental fatigue life data experiencing different failure modes and under other different conditions are used to validate the current methodology, which cover a wide range of metals. A very good agreement is obtained for fatigue life predictions.

The current fatigue model is based on the characteristic plane approach. Most of the existing critical plane-based models can only be applied to certain types of failure modes, i.e. shear-dominated failure or tensile-dominated failure. Their applicability generally depends on the material's properties and loading conditions. In the proposed model, the characteristic plane changes corresponding to different material failure modes, thus making the proposed model have almost no applicability limitation with respect to different metals. The characteristic plane is theoretically determined by minimizing the damage introduced by the hydrostatic strain amplitude. The mean stress effect is also included in the current model through a mean stress effect correction factor.

A useful mechanical parameter is found during the development of the proposed model. The ratio of torsional fatigue strength and uniaxial fatigue strength appears to be very important for the multiaxial fatigue problem.
According to the proposed multiaxial fatigue theory and the experimental data collected in this study, different material failure modes may be related to this parameter. Also, from the experimental results collected from the literature, this parameter shows a good correlation to the extra hardening of the material caused under non-proportional loading.

This paper has considered the applicability of the proposed model to constant amplitude loading. Further work is needed to extend the proposed model to general multiaxial random loading.

REFERENCES


36 ASME Code Case N-47. (1978) Case of ASME Boiler and Pressure Vessel Code, Case N-47, Class 1, Section 3, Division 1, ASME.


