Inverse first-order reliability method for probabilistic fatigue life prediction of composite laminates under multiaxial loading

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Abstract: Multiaxial fatigue reliability is a challenging problem despite extensive progress made during the past few decades. Anisotropic materials, such as composite laminates, are under general multiaxial stress state even if the applied loading is uniaxial. A general methodology for multiaxial fatigue reliability analysis of composite laminates is proposed in this paper. The proposed methodology is based on a unified multiaxial fatigue model for both isotropic and anisotropic materials and the inverse first-order reliability method (Inverse FORM) for probabilistic life prediction. The current fatigue model is a critical plane-based model. The critical plane orientation is theoretically determined by minimizing the damage introduced by the hydrostatic stress amplitude. One of the advantages of the multiaxial fatigue model is that it has almost no applicability limitations with respect to different materials. A time dependent limit state function of material failure is developed based on the proposed mechanism model for probabilistic life prediction. Inverse FORM method is proposed to calculate the fatigue life under a specified failure probability. Various uncertainties from materials properties, ply configurations, and volume fractions are included in the proposed methodology. A wide range experimental fatigue data of composite laminates is used to validate the proposed methodology. It is observed that the proposed methodology gives a satisfactory

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(prediction for both median life and its confidence bounds.

**Keywords:** multiaxial fatigue, composite laminate, reliability, inverse FORM

**Introduction**

Composite materials are widely used for many different industries, such as aerospace and automobile, because of their high strength and stiffness. The long term durability of composite materials is critical for the safety and integrity of structural and mechanical systems. Composite materials are inhomogeneous and anisotropic, which makes the fatigue problems of composite materials more complicated than that of homogenous and isotropic materials (e.g., metallic materials). The fatigue of composite laminates is multiaxial, and a special analysis approach is required for an accurate life prediction. In general, the multiaxial problem can be divided into two cases: one is caused by the anisotropy of composite materials and the other is caused by the external multiaxial loading.

Many engineering materials exhibit some degree of anisotropy in mechanical properties, such as unidirectional and multidirectional composite. Unlike the extensive progress in multiaxial fatigue analysis of isotropic materials, much further effort is needed to include the anisotropy of the material (Miller and Brown, 1985). Several investigations have been reported for anisotropic composite laminates. Degrieck and Van Paepegem (2001) classified existing fatigue models into three categories: fatigue life models (S-N curves), phenomenological models for residual strength or residual stiffness, and progressive damage models. Hasin and Rotem (1973) proposed a failure criterion which mimic the form of static strength criterion, based on two major failure modes (fiber failure and matrix failure). Sims and Brogdon (1977) extended the static failure

Single applied off-axis loading causes proportional multiaxial stress state within the laminates. Most of the fatigue models for anisotropic composite laminates are for proportional multiaxial stress state; however very few theoretical and experimental studies are found in the literature for the non-proportional multiaxial fatigue analysis. No universally accepted multiaxial fatigue damage model exists for different materials and different loading conditions (Liu and Mahadevan, 2007). In addition, huge uncertainties are associated with the fatigue damage process of composite materials, which is usually larger than that of metallic materials (Liu and Mahadevan, 2007). The uncertainties associated with the fatigue damage accumulation can be caused by material properties, structural configurations, and the manufacturing processes. A probabilistic approach is more suitable for fatigue analysis of composite materials.
The key objective of this study is to develop a probabilistic life prediction framework for composite laminates. Many fatigue reliability analysis methodologies used the simulation-based approach, such as direct Monte Carlo simulation to calculate the probabilistic life prediction (Liu and Mahadevan, 2007; Liu and Mahadevan, 2009). This approach is time consuming for system level applications. Other approaches solved the problem in a similar way with that of the time-dependent reliability, i.e., calculating the reliability level at a target life (Liu and Mahadevan, 2007, Liu and Mahadevan, 2009). This approach cannot give direct probabilistic life prediction. A novel inverse first-order reliability method (FORM) is proposed in this study to calculate the residual life directly. The inverse FORM method is originally proposed for reliability-based optimal design (RBDO) problem (Der Kiureghian, Zhang and Li, 1994). Several studies for static failure using the inverse FORM method have been reported in the literature. Sanranyasoonthorn et al. (2004) developed an inverse reliability procedure for wind turbine components. Cheng et al. (2006) applied the inverse FORM method to estimate the cable safety of long-span bridge. Very few studies have been found on the investigation of the inverse FORM method to time dependent fatigue problem, probably due to the difficulties with the implicit response function (Cheng, Zhang, Cai and Xiao, 2007).

In this paper, a general methodology for multiaxial fatigue reliability analysis of composite laminates is proposed. The proposed methodology is based on a unified multiaxial fatigue model for both isotropic and anisotropic materials (Liu and Mahadevan, 2007) and the inverse first-order reliability method (FORM) for time dependent fatigue reliability analysis. The current fatigue model is a critical plane-based model. Most of the earlier models based on the critical plane approach assume that the critical plane only
depends on the stress state. In the current model, the critical plane not only depends on the stress state but also on the material properties. The critical plane is theoretically determined by minimizing the damage introduced by the hydrostatic stress amplitude, which makes the proposed model have almost no applicability limitations with respect to different materials. Various uncertainties from materials properties, ply configurations, and volume fractions are included in the proposed methodology. Probabilistic life prediction using the inverse FORM method is compared with direct Monte Carlo simulation for model verification. A wide range of experimental fatigue data of composite laminates are used to validate the proposed methodology. Generally, the predictions based on the proposed model agree with the experimental observations very well.

**Multiaxial fatigue model**

**Multiaxial fatigue model for isotropic materials**

A multiaxial fatigue damage criterion (Liu and Mahadevan, 2005) was developed based on the nonlinear combination of the normal stress amplitude, shear stress amplitude and hydrostatic stress amplitude acting on the critical plane, as

$$
\sqrt{\left(\frac{\sigma_{a,a}}{f_{-1}}\right)^2 + \left(\frac{\tau_{a,a}}{t_{-1}}\right)^2 + k\left(\frac{\sigma_{a,a}^H}{f_{-1}}\right)^2} = \beta
$$

(1)

where $\sigma_{a,a}$, $\tau_{a,a}$ and $\sigma_{a,a}^H$ are the normal stress amplitude, shear stress amplitude and hydrostatic stress amplitude acting on the critical plane, respectively; $\alpha$ is the angle between the critical plane and the maximum normal stress plane; $f_{-1}$ and $t_{-1}$ are fatigue limits in pure uniaxial and pure shear tests, respectively; and $k$ and $\beta$ are material parameters which can be determined by uniaxial and pure shear fatigue limits. Detailed
derivation and validation of the used multiaxial fatigue model can be found in the
referred article (Liu and Mahadevan, 2005). Only the results of the model parameters are
reported here in Table 1.

In Table 1, \( s = \frac{f_{-\ell}}{f_{-1}} \) is the fatigue strength ratio under the pure shear loading and the
pure uniaxial loading. For any arbitrary multiaxial loading history, the maximum stress
amplitude plane is identified first. This is achieved by enumeration, by changing the
angle by 1 degree increments. Then, the angle \( \alpha \) and material parameters are determined
for different materials according to Table 1. The critical plane is the plane which has an
angle \( \alpha \) with the maximum normal stress amplitude plane. Finally, the stress
components on the critical plane are calculated and the fatigue damage is evaluated using
Eq. (1). Note that the critical plane in the proposed model depends not only on the stress
state (maximum normal stress amplitude plane) but also on the material property (angle
\( \alpha \)).

**Multiaxial Fatigue Model for anisotropic material**

Many engineering materials exhibit mechanical anisotropy, such as wood, rolled
metals, fiber reinforced composite laminates, etc. The uniaxial and torsional fatigue
strengths also depend on the orientations of the axes at the critical point within the
material. In the proposed multiaxial fatigue criterion (Eq. (1)), fatigue limits \( f_{-\ell} \) and \( t_{-\ell} \)
become functions of the orientation \( \theta \), say, \( f_{-\ell}(\theta) \) and \( t_{-\ell}(\theta) \). In order to extend the
fatigue model (Eq. (1)) to anisotropic materials, we need to specify a reference plane, on
which the fatigue strength under uniaxial and pure shear loading can be evaluated. In the
current model, the key point is to calculate the angle between the maximum normal stress
amplitude plane and the critical plane. The reference plane is first defined for the anisotropic material as the plane that experiences the maximum normal stress amplitude. Thus, Eq (1) is rewritten as a unified multiaxial fatigue criterion:

$$\sqrt{\left(\frac{\sigma_{a\alpha}}{f_{-1}(\theta_{\text{max}})}\right)^2 + \left(\frac{\tau_{a\alpha}}{t_{-1}(\theta_{\text{max}})}\right)^2 + k\left(\frac{\sigma_{H\alpha}}{f_{-1}(\theta_{\text{max}})}\right)^2} = \beta$$  \hspace{2cm} (2)

where $\theta_{\text{max}}$ indicates the direction of maximum stress amplitude. For isotropic materials, Eq. (2) reduces to Eq. (1) since the functions $f_{-1}(\theta)$ and $t_{-1}(\theta)$ become constants. The fatigue life model for anisotropic materials can be expressed as:

$$\frac{1}{\beta} \sqrt{\left(\frac{\sigma}{s_{N_f}(\theta_{\text{max}})}\right)^2 + \left(\frac{1}{p_{N_f}(\theta_{\text{max}})}\right)^2 (\tau_{a\alpha})^2 + k(\sigma_{a\alpha})^2} = f_{N_f}(\theta_{\text{max}})$$  \hspace{2cm} (3)

Eq. (3) can be rewritten as:

$$\frac{1}{p_{N_f}(\theta_{\text{max}})} \frac{1}{\beta} \sqrt{\left(\frac{\sigma}{s_{N_f}(\theta_{\text{max}})}\right)^2 + \left(\frac{1}{p_{N_f}(\theta_{\text{max}})}\right)^2 (\tau_{a\alpha})^2 + k(\sigma_{a\alpha})^2} = f_{N_f}(0)$$  \hspace{2cm} (4)

where $s_{N_f}(\theta_{\text{max}}) = \frac{t_{N_f}(\theta_{\text{max}})}{f_{N_f}(\theta_{\text{max}})}$ is the strength ratio of under pure shear loading and the uniaxial loading along the direction of $\theta_{\text{max}}$. $p_{N_f}(\theta_{\text{max}}) = \frac{f_{N_f}(\theta_{\text{max}})}{f_{N_f}(0)}$ is the ratio of uniaxial strength along the directions of $\theta = \theta_{\text{max}}$ and $\theta = 0$. The left side of Eq. (4) can be treated as an equivalent stress amplitude. It can be used to correlate with the fatigue life using the uniaxial S-N curve along the direction of zero degree. Detailed derivation and concept can be found in (Liu and Mahadevan,2007)

The procedure for the fatigue analysis of anisotropic materials is almost identical with that of isotropic material. For any arbitrary loading history, the maximum stress
amplitude plane is identified first. The uniaxial and pure shear fatigue strength along this direction is also evaluated from experimental data. Then the angle $\alpha$ and the material parameters are determined for different materials according to Table 1. Notice that, the quantity $s$ in Table 1 is now redefined as $s = s_{N_f}(\theta_{max}) = \frac{f_{N_f}(\theta_{max})}{\theta_{max}}$. Finally the equivalent stress amplitude and the fatigue life are calculated using Eq. (4).

For an arbitrary anisotropic material, the variation of the uniaxial and pure shear fatigue strengths corresponding to the orientation of the axes is quite complex and requires extensive experimental work to quantify. However, for some special anisotropic materials, this can be simplified using one of the strength theories available in the literature. In this paper, an example of orthotropic composite laminate is used for illustration.

Consider a fiber reinforced composite laminate. Several static strength theories have been proposed for orthotropic laminates, such as Tsai-Hill and Tsai-Wu theory (Daniel and Ishai, 2006). In this study, the Tsai-Wu theory is used. For the case of plane stress, the Tsai-Wu theory is expressed as:

$$F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\sigma_6^2 + F_1\sigma_1 + F_2\sigma_2 + 2F_{12}\sigma_1\sigma_2 = 1$$

(5)

where $\sigma_1$ and $\sigma_2$ are the stresses along the fiber direction and transverse to the fiber direction, respectively, and $\sigma_6$ is the in-plane shear stress. $F_{11}$, $F_{22}$, $F_{66}$, $F_{12}$, $F_1$, and $F_2$ are strength parameters and can be calibrated using experiments.

$$\begin{cases} F_{11} = \frac{1}{s_L^+s_L^-}, & F_1 = \frac{1}{s_L^+} - \frac{1}{s_L^-}, & F_{22} = \frac{1}{s_T^+s_T^-}, & F_2 = \frac{1}{s_T^+} - \frac{1}{s_T^-} \\ F_{66} = \frac{1}{s_{LT}^+}, & F_{12} = -\frac{(F_{11}F_{22})^{0.5}}{2} \end{cases}$$

(6)
where \( s_L^{(\pm)}, s_T^{(\pm)} \) are the strengths along the fiber direction and transverse to the fiber direction, respectively. The plus symbol indicates tension strength and the minus symbol indicates compression strength. \( s_{LT} \) is the in-plane shear strength. For the fatigue problem, the stress terms in Eq. (5) refer to the stress amplitudes along different directions. If the strengths are defined using stress amplitude values, the plus and minus symbols in the above strength notation disappear since the stress amplitude is always positively defined. Thus, Eq. (5) and Eq. (6) are rewritten for the fatigue problem as:

\[
F_{11}\sigma_{1}^2 + F_{22}\sigma_{2}^2 + F_{66}\sigma_{6}^2 + 2F_{12}\sigma_{1}\sigma_{2} = 1
\]

(7)

\[
\begin{align*}
F_{11} &= \frac{1}{s_L^2}, & F_{22} &= \frac{1}{s_T^2}, & F_{66} &= \frac{1}{s_{LT}^2}, & F_{12} &= -\frac{(F_{11}F_{22})^2}{2} \\
\end{align*}
\]

(8)

Using the Tsai-Wu strength theory, the uniaxial strength and shear strength along an arbitrary direction \( \theta \) can be easily obtained as

\[
\begin{align*}
\begin{bmatrix}
f(\theta) \\
t(\theta)
\end{bmatrix} &= \frac{1}{\sqrt{F_{11}\cos^4\theta + F_{22}\sin^4\theta + (F_{66} + 2F_{12})\sin^2\theta \cos^2\theta}} \\
&= \frac{1}{\sqrt{(F_{11} + F_{12} - 8F_{12})\sin^2\theta \cos^2\theta + F_{66}(\cos^2\theta - \sin^2\theta)^2}}
\end{align*}
\]

(9)

For the fatigue life model, the fatigue strength coefficients are also functions of the fatigue life \( (N_f) \), which can be evaluated from the experimental S-N curves. Eq. (9) is rewritten as:

\[
\begin{align*}
\begin{bmatrix}
f_{N_f}(\theta) \\
t_{N_f}(\theta)
\end{bmatrix} &= \frac{1}{\sqrt{F_{11,N_f}\cos^4\theta + F_{22,N_f}\sin^4\theta + (F_{66,N_f} + 2F_{12,N_f})\sin^2\theta \cos^2\theta}} \\
&= \frac{1}{\sqrt{(F_{11,N_f} + F_{12,N_f} - 8F_{12,N_f})\sin^2\theta \cos^2\theta + F_{66,N_f}(\cos^2\theta - \sin^2\theta)^2}}
\end{align*}
\]

(10)

Substituting Eq. (10) into Eq. (4), we can solve for the fatigue life \( (N_f) \). Similar to isotropic materials, Eq. (4) usually has no closed form solution. In practical calculation, a trial and error method can be used to find \( N_f \). For an orthotropic composite laminate, the
experimental S-N curves along the fiber direction, transverse to the fiber direction, and in-plane shear stress are required in the proposed model. Then the fatigue life under arbitrary multiaxial loading can be predicted.

The fatigue model for the isotropic material is consistent with the fatigue model for the anisotropic material derived in this section. If $F_{11} = F_{22} = \frac{1}{3} F_{66}$, the fatigue model for the orthotropic material is identical with the fatigue model for the isotropic material with $s = \sqrt{3}$, in which the Tsai-Wu criterion reduces to the von Mises criterion.

The above discussion can be easily applied to a laminate with multiple plies, following the steps described in (Liu and Mahadevan, 2005). First, divide the total fatigue life into several blocks. In each block, check the failure of each ply using the above model. If no failure occurs, accumulate the fatigue damage for each ply. If failure occurs, assume that the ply strength and stiffness decrease to zero. Then update the global stiffness matrix and proceed to the next step. The computation continues till the entire laminate fails. The number of the loading cycles to failure is the fatigue life of the composite laminate.

**Inverse FORM method**

The above discussion is for deterministic analysis and is not sufficient to capture the stochastic behavior of fatigue damage of composite materials. A general inverse reliability methodology is proposed in this study to include various uncertainties from materials, geometries and manufacturing for probabilistic fatigue life prediction of unidirectional and multidirectional composite laminates. Details are shown below.

**Inverse FORM method**

The first-order reliability method is a widely used numerical technique to calculate the reliability or failure probability of various engineering problems. Many studies have been
reported on static failure problems using the FORM method (Thorndahl and Willems, 2008, Skaggs and Barry, 1996, Cizelj, Mavko and Riesch-Oppermann, 1994). It has been applied to fatigue problems to calculate the time dependent reliability. Unlike the FORM method (Liu and Mahadevan, 2009, Haldar and Mahadevan, 2000) the inverse FORM method tries to solve the unknown parameters under a specified reliability or failure probability level, which is more suitable for probabilistic life prediction (i.e., remaining life estimation corresponding to a target reliability level). In the inverse FORM method, a limit state function needs to be developed first, such as the generic expression of Eq. (11-a). \( x \) is the vector of random variables and \( y \) is the vector of indexing variables. For example, \( x \) could be the random material properties, loadings, and environmental factors and \( y \) could be the time and spatial coordinates. The limit state function need be transformed to the standard normal space for the calculation, which is similar to the classical FORM method (Haldar and Mahadevan, 2000). The numerical search for the unknown parameters needs to satisfy the reliability constraints, which are described in Eqs. 11(b-c). \( \beta \) is the reliability index, which is defined as the distance from origin to the most probable point (MPP) in the standard normal space. The failure probability \( P_f \) can be calculated using the cumulative distribution function (CDF) \( \Phi \) of the standard Gaussian distribution. Numerical search is required to find the optimum solution, which satisfies the limit state function (Eq. 11(d)). Details of the general inverse FORM method and concept can be found in (Der Kiureghian, Zhang and Li, 1994).
The overall objective of the inverse FORM method is to find a non-negative function satisfying all constraint conditions specified in Eq. (11). Then the numerical search algorithm can be used to find the solutions of the unknown parameters. Following the general concept of the first-order reliability method, the limit state function is approximated using the first-order Taylor’s series expansion to facilitate the calculation. First, the limit state function Eq. 11(a) is expanded around random variable vector \( x \) and the indexing variable vector \( y \) is fixed.

\[
g(x, y) = g(\mu_{x_1}, \mu_{x_2}, ..., \mu_{x_n}) + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(x - \mu_{x_i}) + O(\mu_{x_1}, \mu_{x_2}, ..., \mu_{x_n}) = 0 \tag{12}
\]

Eq. (12) can be rewritten as

\[
x = \frac{\nabla_x g(u, y) \cdot u - g(u, y)}{\|\nabla_x g(u, y)\|^2} \nabla_x g(u, y) \tag{13}
\]

where

\[
\nabla_x g(u, y) = \left( \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, ..., \frac{\partial g}{\partial x_n} \right)
\]

The increments of \( x \) and \( y \) can be expressed as

\[
\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\nabla_x g(x, y) \cdot x - g(x, y)}{\|\nabla_x g(x, y)\|^2} \nabla_x g(x, y) - x \\ 0 \end{bmatrix} \tag{14}
\]

A non-negative merit function considering the constraints of Eq. 11(a) and Eq. 11(d) can be written as
In Eq.(15), both $k_1$ and $k_2$ are constants. Next, the reliability constraint (Eq. 11(b)) needs to be included. Substitute Eq.(11-b) into Eq.(11-d), one can obtain

$$x = -\beta_{\text{target}} \frac{\nabla_x g(x, y)}{\|\nabla_x g(x, y)\|} \quad (16)$$

Using first order Taylor’s series expression, the limit state function can be expanded around $x$ and $y$ as

$$g(x, y) = g(\mu_{x_1}, \mu_{x_2}, \ldots, \mu_{x_n}, \mu_{y_1}, \mu_{y_2}, \ldots, \mu_{y_m}) + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} (x_i - \mu_{x_i}) + \sum_{j=1}^{m} \frac{\partial g}{\partial y_j} (y_j - \mu_{y_j}) + O(\mu_{x_1}, \mu_{x_2}, \ldots, \mu_{x_n}, \mu_{y_1}, \mu_{y_2}, \ldots, \mu_{y_m}) = 0 \quad (17)$$

Substitute Eq.(17) into Eq.(18), the indexing variables can be expressed as

$$y = \mu_y + \frac{\nabla_x g(x, \alpha) \cdot \mathbf{x} - g(x, y) + \beta_{\text{target}} \|\nabla_x g(x, y)\|}{\frac{\partial g(x, y)}{\partial y}} \quad (18)$$

The increments of $x$ and $y$ can be expressed as

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} -x - \beta_{\text{target}} \frac{\nabla_x g(x, y)}{\|\nabla_x g(x, y)\|} \\ \nabla_x g(x, y) \cdot \mathbf{x} - g(x, y) + \beta_{\text{target}} \|\nabla_x g(x, y)\| \frac{\partial g(x, y)}{\partial y} \end{bmatrix} \quad (19)$$

A merit function considering the reliability constraints can be written as

$$f^2(x, \alpha) = \frac{1}{2} k_s (\|\mathbf{x} \| - \beta_{\text{target}})^2 \quad (20)$$

Combine the two merit function Eq. (15) and Eq. (20), a general function is obtained as
\[
f(x, y) = f^1(x, y) + f^2(x, y)
\]
\[
= \frac{1}{2} k_1 \left\| x - \frac{[\nabla_x g(x, y) \cdot x]}{\|\nabla_x g(x, y)\|^2} \nabla_x g(x, y) \right\|^2 + \frac{1}{2} k_2 g(x, y)^2 + \frac{1}{2} k_3 (\|x\| - \beta)^2
\]  \hspace{1cm} (21)

Numerical search algorithm is developed to iteratively solve the Eq. (21). The search algorithm is expressed as Eq. (22) after \( k \) iterations.

\[
\begin{align*}
X_{k+1} &= X_k + d_k = X_k + (a_1 d^1_k + a_2 d^2_k) \\
y_{k+1} &= y_k + (x_{k+1} - x_k)
\end{align*}
\]  \hspace{1cm} (22)

where \( d^1_k \) and \( d^2_k \) are the search directions corresponding to different merit functions and can be calculated using Eq. (14) and Eq. (19), respectively. \( a_1 \) and \( a_2 \) are the weight factors and can be calculated as

\[
\begin{align*}
a_1 &= \frac{f^1(x)}{f^1(x) + f^2(x)} \\
a_2 &= \frac{f^2(x)}{f^1(x) + f^2(x)}
\end{align*}
\]  \hspace{1cm} (23)

The convergence criterion for the numerical search algorithm is

\[
\frac{(\|x_{k+1} - x_k\|^2 + \|y_{k+1} - y_k\|^2)^{\frac{1}{2}}}{(\|x_{k+1}\|^2 + \|y_{k+1}\|^2)^{\frac{1}{2}}} \leq \varepsilon
\]  \hspace{1cm} (24)

where \( \varepsilon \) is a small value and indicates that the relative difference between two numerical solutions is small enough to ensure the convergence. Using the proposed methodology, the complex probabilistic fatigue life prediction problem can be solved efficiently compared to the direct Monte Carlo simulation method. It is noted that the above derivation assumes the random variables are standard Gaussian variables. In practical engineering application, non-Gaussian variables are commonly used for some non-negative physical quantities, such as strength and Young’s modulus. The proposed
inverse FORM method can be extended to non-Gaussian variables with proper random variable transformation. This paper uses the transformation method proposed by Rackwitz and Fiessler (June 1976) to transform the non-Gaussian variables to their equivalent standard normal space. After that the proposed inverse method can be used. Once the solutions are obtained in the standard normal space, the inverse transformation can be used to transform the solution to its original space. The random variable transformation can be expressed as

\[
\begin{align*}
\Phi\left(\frac{x^* - \mu_X^N}{\sigma_X^N}\right) = F_X(x^*) &\Rightarrow \mu_X^N = x^* - \Phi^{-1}\left[F_X(x^*)\right]\sigma_X^N \\
\frac{1}{\sigma_X^N} \phi\left(\frac{x^* - \mu_X^N}{\sigma_X^N}\right) = f_X(x^*) &\Rightarrow \sigma_X^N = \phi\left[\Phi^{-1}\left[F_X(x^*)\right]\right] \left(\frac{1}{f_X(x^*)}\right)
\end{align*}
\]

(25)

where \(\Phi(\cdot)\) and \(F_X(x^*)\) are the cumulative distribution functions (CDF) of the standard normal random variable and the non-normal random variable, respectively. \(\phi(\cdot)\) and \(f_X(x^*)\) are the probability density function (PDF) of the standard normal random variable and the non-normal random variable, respectively. This transformation works well for the fatigue problem of composite laminates since the distributions of random variables are not highly skewed. For highly skewed distribution, the transformation proposed by Rackwitz and Fiessler (1978) can be used instead.

**Numerical example and model verification**

The above discussed inverse FORM method is applied to probabilistic fatigue life prediction of composite laminates integrating the mechanism model. The limit state function is shown as

\[
g() = f(X_1, X_2, \ldots, X_{13}) - N = 0
\]

(26)
$X_{1,13}$ are the random variables. $N$ is the indexing factor and represents the failure time. $f(\cdot)$ represents the proposed model of fatigue life prediction for composite materials. It is noted that the $f(\cdot)$ is a generic implicit function and no analytical solution for the derivatives is available. The perturbation-based finite difference method (Sauer, 2006) is used to calculate the first-order derivatives in the proposed inverse FORM framework. Thirteen random variables are included in the calculation. They include the elastic modulus ($E_1, E_2$), Poisson’s ratio ($\nu_{12}$), shear modulus ($G_{12}$), volume fraction of fibers ($V_f$), ply thickness ($t$), and ply orientations ($\theta$). These random variables represent the basic material properties, geometric configurations, and manufacturing factors. Material random fatigue properties are also included. The fitting parameters of the material S-N curves are assumed to be random variables. The classical power function is used to describe the fatigue S-N curves under longitudinal, transverse, and pure shear loadings, i.e.,

$$S = A \cdot N^B$$

where $S$ is the stress amplitude level and $N$ is the fatigue life. $A$ and $B$ are material properties and are assumed to be random variables. Since three independent fatigue S-N curves are required in the proposed mechanism model, six random variables are included. A D155 balanced laminate is selected for numerical example, which consists of three pairs of ply with identical thickness and elastic properties but with $\pm 10$ degree orientations. The mean value of the above mentioned random variables can be found in (Mandell J.F., Feb, 2003.). For demonstration purpose, the coefficient of variation for all random variables is assumed to be 0.05. All random variables are assumed to be
lognormal variables except for the power coefficient $B$ in Eq. (24), which can take the negative value and is assumed to follow normal distribution.

Fig.1 Comparison of the direct MC method with the inverse FORM method

Fig.1 shows the probabilistic life prediction using both inverse FORM method and the direct Monte Carlo method. The solid line is the result of Monte Carlo (MC) Simulation with one million samples at a certain stress level at 122 MPa. The inverse FORM results are shown as the triangular points, which agree well with the MC simulation. The proposed inverse FORM can give an accurate result and significantly reduce the computational time. It takes 29044 seconds using the MC simulation. The inverse FORM method takes 422 seconds. All computations are performed using Matlab 2007 on a dual-core PC (2.66 GHz) with 3 Gb memory. The operating system is Windows XP Professional.

Validation of the proposed method

A wide range of experimental data on unidirectional and multidirectional composite laminates is used to demonstrate and validate the proposed probabilistic life prediction methodology.

Experimental data and statistics of input random variables

Seven sets of fatigue experimental data for unidirectional composite laminate under off-axis loading are employed in this section, and are listed in Table 2.

The experimental S-N curves along the fiber direction, transverse to the fiber direction, and pure in-plane shear stress are required in the proposed fatigue model. The curves along and transverse to the fiber direction are usually reported. However, most of the fatigue experimental data do not include the pure shear test results. It is possibly due to
the difficulty of applying the pure shear loading to the composite laminate. In the proposed study, the S-N curve under pure in-plane shear stress is calibrated using one additional off-axis fatigue test data by a trial and error method (Liu and Mahadevan, 2005). For example, the S-N curves for a D155 balanced laminate along the fiber and transverse to the fiber are reported in (Mandell J.F., Feb, 2003.). The experimental data is shown in Fig. 2 (a) and (b) for 0° and 90° respectively. Statistical analysis can be done and the distribution of $\text{Along, Blong, Atran, Btran}$ can be obtained. However, no experimental data were reported under pure shear loading to obtain $\text{Ashear or Bshear}$ directly. The pure shear S-N curve is calibrated using the balanced laminate ($\pm45^\circ$) and $\text{Ashear and Bshear}$ can be obtained. Once the S-N curves are obtained, the fatigue life of composite laminates can be predicted for arbitrary orientations. The mean values of the strength coefficients are shown in Table 3. All the six strength coefficients are the input random variables for both unidirectional materials. All the input random variables are assumed to follow log-normal distribution, except $\text{Blong, Btran and Bshear}$, which follow normal distribution.

Fig. 2 Experimental data: (a) ($\pm0\_3$), (b) ($\pm90\_3$)

For multidirectional material, another seven input random variables, the elastic modulus ($E_1, E_2$), Poisson’s ratio ($\nu_{12}$), shear modulus ($G_{12}$), volume fraction of fibers ($V_f$), ply thickness ($t$), and ply orientations ($\theta$) are included in the current model for calculation. The geometry properties and volume fraction of the balanced laminate D155 are reported in (Mandell J.F., Feb, 2003.). The coefficient of variation of all the other five random variables are assumed to be 0.05 (Liu and Mahadevan, 2005).

**Validation for unidirectional composite materials**
In Fig. 3, the model prediction of the median life and its 90% confidence bounds are plotted together with the experimental data. The $x$-axis is the fatigue life and the $y$-axis is the stress amplitude. A semi-log scale plot is used (i.e., only the $x$-axis is in log scale). The solid lines are the median prediction results and dashed lines are the 90% confidence bounds. All points are the experimental observations under off-axis loading at different angles. The angles of the off-axis loading are shown in the legends. As shown in Fig. 3, the median prediction results agree very well with the experimental results. In addition, different uncertainties of experimental data can be quantitatively predicted using the proposed probabilistic methods. Almost all the experimental data lies in the 90% confidence bounds.

Fig. 3 Comparison of life prediction with experimental data for unidirectional composite laminates

Validation for multidirectional composite laminates

Fatigue test data of glass-fiber-based multidirectional composite laminates (Mandell J.F., Feb, 2003.) are used to validate the proposed fatigue model. The material chosen, D155, is a balanced laminate which consists of pairs of layers with identical thicknesses and elastic properties but with $\pm 20^\circ$, $\pm 30^\circ$, $\pm 40^\circ$, $\pm 50^\circ$, $\pm 60^\circ$, $\pm 70^\circ$, $\pm 80^\circ$. Again, the fatigue S-N curve for pure shear test is not available. In the current study the balanced laminate ([±45]$_3$) is used to calibrate the shear S-N curve.

The prediction results and the experimental observations are plotted in Fig. 4. The $x$-axis is the fatigue life and the $y$-axis is the applied stress amplitude. The solid lines are the median prediction results and dashed lines are the 90% confidence bounds. From Fig. 4, generally satisfactory results can be observed with a few exceptions. In all cases, the
median predictions capture the major trends in the experimental observations. The 90% confidence bounds covers majority of the experimental data.

Fig. 4 Comparison of life prediction with experimental data for multidirectional composite laminates

Conclusion

A general probabilistic life prediction methodology is proposed in this paper combining a critical plane-based multiaxial model and the inverse first-order reliability method. The multiaxial fatigue model can be applied to both isotropic and anisotropic materials. The proposed inverse FORM method can efficiently calculate the fatigue life prediction corresponding to different target reliability level compared to the direct Monte Carlo simulation method. Several conclusions can be drawn based on the current investigations.

- Overall satisfactory results are observed between model predictions and experimental results for both unidirectional and multidirectional composite laminates.

- The proposed inverse FORM method has been verified with direct Monte Carlo simulation results and validated with extensive experimental data.

- The scatter of experimental data can be predicted using the quantified uncertainties of material properties, geometric configurations, and manufacturing processes and the proposed probabilistic framework.

- It is observed that the predictions results have a better agreement for unidirectional composite laminates, which suggests that a more comprehensive mechanism model for multidirectional composite laminates is required to include
other factors, such as delaminating between plies.

Current investigation focuses on the constant proportional multiaxial loading.

Further model development and validation are needed for general nonproportional random loading. Geometric effects, such as holes and notches, need further study for structural level applications.

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The research reported in this paper was supported by funds from NSF (Award No. CMMI-0900111, Project Manager: Dr. Mahendra Singh) and by funds from National Aeronautics and Space Administration (NASA) (Contract No. NNX09AY54A, Project Manager: Dr. Kai Goebel). The support is gratefully acknowledged.
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Fig. 3 Comparison of life prediction with experimental data for unidirectional composite laminates (a) AS4/PEEK; (b) E-glass fibre/exopy-1 with R = 0; (c) E-glass fibre/exopy-1 with R = 0.5; (d) E-glass fibre/exopy-1 with R = -1; (e) T800H/2500 carbon/epoxy with R = 0.1; (f) T800H/2500 carbon/epoxy with R = -0.3; (g) T800H/2500 carbon/epoxy with R = 0.5; (h) E-glass/polyester; (i) T800H/epoxy; (j) T800H/polymide; (k) GLARE 2 (l) GLARE 2

Fig. 4 Comparison of life prediction with experimental data for multidirectional composite laminates (a) ±20°; (b) ±30°; (c) ±40°; (d) ±50°; (e) ±60°; (f) ±70°; (g) ±80°

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Table 2. Experimental data for unidirectional materials

Table 3. Experimental data for uniaxial materials and multidirectional material
### Table 1. Material parameters for fatigue damage evaluation

<table>
<thead>
<tr>
<th>Material Property</th>
<th>$s = \frac{t_{-1}}{f_{-1}} \leq 1$</th>
<th>$s = \frac{t_{-1}}{f_{-1}} &gt; 1$</th>
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<tbody>
<tr>
<td>$\alpha$</td>
<td>$\cos(2\alpha) = \frac{-2 + \sqrt{4 - 4(1/s^2 - 3)(5 - 1/s^2 - 4s^2)}}{2(5 - 1/s^2 - 4s^2)}$</td>
<td>$\alpha = 0$</td>
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<tr>
<td>$k$</td>
<td>$k = 0$</td>
<td>$k = 9(s^2 - 1)$</td>
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<tr>
<td>$\beta$</td>
<td>$\beta = (\cos^2(2\alpha)s^2 + \sin^2(2\alpha))^\frac{1}{2}$</td>
<td>$\beta = s$</td>
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### Table 2. Experimental data for unidirectional materials

<table>
<thead>
<tr>
<th>Material</th>
<th>References</th>
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<tr>
<td>E-glass/polyester</td>
<td>Philippidis and Vassilopoulos (Sep 1999)</td>
</tr>
<tr>
<td>E-glass fibre/epoxy-1</td>
<td>Kadi and Ellyin (1994)</td>
</tr>
<tr>
<td>T800H/epoxy</td>
<td>Kawai et al (2001)</td>
</tr>
<tr>
<td>T800H/polyimide</td>
<td>Kawai et al (2001)</td>
</tr>
<tr>
<td>GLARE 2(fibre–metal laminates)</td>
<td>Kawai et al (2001)</td>
</tr>
<tr>
<td>T800H/2500 carbon/epoxy</td>
<td>Kawai and Suda (May 2004)</td>
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Table 3. Experimental data for uniaxial materials and multidirectional material

<table>
<thead>
<tr>
<th>Materials \ Random variable</th>
<th>Along (MPa)</th>
<th>Blong (MPa)</th>
<th>Atran (MPa)</th>
<th>Btran (MPa)</th>
<th>Ashear (MPa)</th>
<th>Bshear</th>
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<td>AS4/PEEK</td>
<td>1903.9</td>
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<td>E-glass fibre/epoxy-1 (R=0)</td>
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<td>E-glass fibre/epoxy-1 (R=0.5)</td>
<td>203.49</td>
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<td>11.683</td>
<td>-0.0269</td>
<td>13.892</td>
<td>-0.0206</td>
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<tr>
<td>E-glass fibre/epoxy-1 (R=-1)</td>
<td>810.06</td>
<td>-0.0603</td>
<td>43.44</td>
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<td>-0.0861</td>
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<td>1111.1</td>
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<td>T800H/2500 carbon/epoxy (R=0.3)</td>
<td>591.55</td>
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<td>-0.05</td>
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<td>-0.07</td>
<td>59</td>
<td>-0.0553</td>
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<td>GLARE 2(fibre–metal laminates)</td>
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<td>D155</td>
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<td>72.485</td>
<td>-0.0469</td>
<td>139.16</td>
<td>-0.0633</td>
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